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orbit of the planet depends. Equation (8) shows that if two of the distances are known the third can be computed. Equation (5) furnishes the relation between the curtate distances used by Olbers in his well known method of computing the orbit of a comet.

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SOLUTIONS OF EXERCISES.

10

REQUIRED the length of a thread wrapped spirally round the frustum of a given cone, the distance between the spires along the slant height being constant.

[*A. B. Nelson.*]

SOLUTION.

The development of the thread on a plane obtained by rolling the cone on the plane will be a spiral of Archimedes, the length of which is well known.

[*De Volson Wood.*]

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THE RESULT

$$\frac{p^2q^2 + 4p^3r - 8q^3 + 2pqr + 9r^3}{(r - pq)^2}$$

is given as the equivalent of the function

$$\left(\frac{\beta - \gamma}{\beta + \gamma}\right)^2 + \left(\frac{\gamma - \alpha}{\gamma + \alpha}\right)^2 + \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2,$$

where α, β, γ are the roots of the cubic

$$x^3 + px^2 + qx + r = 0.$$

Is this result correct? [*A. Hall.*]

SOLUTION.

The result is *not* correct. The symmetric function $\Sigma\left(\frac{\beta - \gamma}{\beta + \gamma}\right)^2$ of the roots α, β, γ of the cubic $x^3 + px^2 + qx + r = 0$, expressed in terms of the co-efficients, is

$$= \frac{-3p^2q^2 + 4p^3r + 4q^3 + 2pqr + 9r^2}{(r - pq)^2}.$$

Indeed, we have

$$\Sigma\left(\frac{\beta-\gamma}{\beta+\gamma}\right)^2=\frac{\Sigma(\beta-\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2}{(\beta+\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2},$$

say $=\frac{N}{\Delta}$, and on the other hand the well known relations $\Sigma\alpha=-p$, $\Sigma\beta\gamma=q$, $\alpha\beta\gamma=-r$, where Σ always indicates summation with respect to the three roots.

For the denominator Δ we find:

$$\begin{aligned}(\beta+\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2&=(\Sigma\alpha-\alpha)^2(\Sigma\alpha-\beta)^2(\Sigma\alpha-\gamma)^2\\&=(p+\alpha)^2(p+\beta)^2(p+\gamma)^2\\&=(p^3+\Sigma\alpha\,.\,p^2+\Sigma\alpha\beta\,.\,p+\alpha\beta\gamma)^2\\&=(r-pq)^2.\end{aligned}$$

To express the numerator N in terms of p, q, r, we observe first that the "weight" of the function is 6. The general form of the expression will therefore be

$$Ap^2q^2+Bp^3r+Cq^3+Dpqr+Er^2,$$

since this contains all possible combinations of p, q, r of the weight 6. This agrees with the general form of the expression as stated in the problem, with the exception of the last term, where r^3 is probably a misprint for r^2.

The most expedient way to determine the numerical co-efficients seems to be the following: —

Assume any five cubic equations whose roots are known, and calculate for every one the values of $\Sigma(\beta-\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2$ and of the co-efficients of A, B, C, D, E in the general form. This gives five equations sufficient for the determination of A, B, C, D, E.

Taking for instance the equations

$$\begin{aligned}0&=(x-1)^2(x+2)&&=x^3-3x+2, &&(1)\\0&=(x-1)(x-2)(x+3)&&=x^3-7x+6, &&(2)\\0&=(x-1)^2(x+1)&&=x^3-x^2-x+1, &&(3)\\0&=(x-1)^3&&=x^3-3x^2+3x-1, &&(4)\\0&=(x-1)(x-2)(x-3)&&=x^3-6x^2+11x-6; &&(5)\end{aligned}$$

we have

$\alpha=$	$\beta=$	$\gamma=$	$p=$	$q=$	$r=$	
$+1$	$+1$	-2	0	-3	$+2$	(1)
$+1$	$+2$	-3	0	-7	$+6$	(2)
$+1$	$+1$	-1	-1	-1	$+1$	(3)
$+1$	$+1$	$+1$	-3	$+3$	-1	(4)
$+1$	$+2$	$+3$	-6	$+11$	-6	(5)

Substituting in the equation

$$Ap^2q^2 + Bp^3r + Cq^3 + Dpqr + Er^2 = \Sigma(\beta - \gamma)^2(\gamma + \alpha)^2(\alpha + \beta)^2,$$

we have

$$\begin{array}{rrrrrl}
 & & -\ 27C & & +\ 4E = & +\ 72, \\
 & & -\ 343C & & +\ 36E = & +\ 1048, \\
+A - & B - & C + & D + & E = & 0, \\
+81A + & 27B + & 27C + & 9D + & E = & 0, \\
+4356A + & 1296B + & 1331C + & 396D + & 36E = & +\ 1444;
\end{array}$$

from which we find

$$A = +3, B = -4, C = -4, D = -2, E = -9.$$

This gives

$$\Sigma(\beta - \gamma)^2(\gamma + \alpha)^2(\alpha + \beta)^2 = +3p^2q^2 - 4p^3r - 4q^3 - 2pqr - 9r^2.$$

The direct derivation of this expression from the fundamental relations $\Sigma\alpha = -p$, $\Sigma\beta\gamma = q$, $\alpha\beta\gamma = -r$ is somewhat lengthy. It may be effected in the following way:—

$$\begin{aligned}
\Sigma(\beta - \gamma)^2(\gamma + \alpha)^2(\alpha + \beta)^2 &= \Sigma(\beta - \gamma)^2(\alpha^2 + \beta\gamma + \gamma\alpha + \alpha\beta)^2 \\
&= \Sigma(\beta - \gamma)^2(\alpha^2 + q)^2 \\
&= \Sigma\alpha^4(\beta - \gamma)^2 + 2q\,\Sigma\alpha^2(\beta - \gamma)^2 + q^2\,\Sigma(\beta - \gamma)^2
\end{aligned}$$

To calculate these three terms we want the following relations:—

$$\begin{aligned}
(\Sigma\alpha)^2 &= p^2 = \Sigma\alpha^2 + 2\Sigma\beta\gamma, \\
\therefore\ \Sigma\alpha^2 &= p^2 - 2q; \\
(\Sigma\beta\gamma)^2 &= q^2 = \Sigma\beta^2\gamma^2 + 2\Sigma\alpha^2\beta\gamma, \\
\therefore\ \Sigma\beta^2\gamma^2 &= q^2 - 2pr; \\
\Sigma\alpha^2.\Sigma\beta^2\gamma^2 &= (p^2 - 2q)(q^2 - 2pr) \\
&= \Sigma\alpha^4(\beta^2 + \gamma^2) + 3\alpha^2\beta^2\gamma^2, \\
\therefore\ \Sigma\alpha^4(\beta^2 + \gamma^2) &= p^2q^2 - 2q^3 - 2p^3r + 4pqr - 3r^2; \\
\Sigma\alpha\,.\,\Sigma\beta\gamma &= -pq = \Sigma\beta^2\gamma + 3\alpha\beta\gamma, \\
\therefore\ \Sigma\beta^2\gamma &= 3r - pq; \\
\Sigma\alpha\,.\,\Sigma\alpha^2 &= -p(p^2 - 2q) = \Sigma\alpha^3 + \Sigma\beta^2\gamma, \\
\therefore\ \Sigma\alpha^3 &= -p^3 + 3pq - 3r; \\
\alpha\beta\gamma\,.\,\Sigma\alpha^3 &= -r(3pq - p^3 - 3r) = \Sigma\alpha^4\beta\gamma \\
&= -3pqr + p^3r + 3r^2.
\end{aligned}$$

With the aid of these expressions we find:—

$$\Sigma(\beta - \gamma)^2 = 2\Sigma\alpha^2 - 2\Sigma\beta\gamma = 2(p^2 - 3q),$$

$$\Sigma\alpha^2(\beta-\gamma)^2 = 2\,\Sigma\beta^2\gamma^2 - 2\,\Sigma\alpha^2\beta\gamma = 2\,(q^2 - 3pr),$$

$$\begin{aligned}\Sigma\alpha^4(\beta-\gamma)^2 &= \Sigma\alpha^4(\beta^2+\gamma^2) - 2\Sigma\alpha^4\beta\gamma\\ &= p^2q^2 - 2q^3 - 4p^3r + 10pqr - 9r^2;\end{aligned}$$

$$\therefore\ \Sigma\,(\beta-\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2 = +\,3p^2q^2 - 4p^3r - 4q^3 - 2pqr - 9r^2,$$

as above.

[*Alexander Ziwet.*]

14

A FAISCEAU of parabolas can be drawn having the pole of a cardioid as the common focus, all passing through one point and all cutting the cardioid at right angles. [*H. A. Newton.*]

SOLUTION.

The equation of a cardioid is

$$r = m\,(1 - \cos\theta);$$

and that of a parabola whose focus is at the pole of the cardioid is

$$r = \frac{p}{1 - \cos(\theta - a)}.$$

When these equations are differentiated the former yields

$$r\,\frac{d\theta}{dr} = \frac{1 - \cos\theta}{\sin\theta},$$

and the latter

$$r\,\frac{d\theta}{dr} = -\,\frac{1 - \cos(\theta - a)}{\sin(\theta - a)},$$

as expressions for the tangents of the angles made by the respective curves with the radius vector. Hence, if θ' represent the vectorial angle belonging to that radius vector which meets the two curves at angles differing by 90°, we have

$$\frac{1 - \cos\theta'}{\sin\theta'} \cdot \frac{1 - \cos(\theta' - a)}{\sin(\theta' - a)} = 1\,;$$

and if the same radius vector meet the curves at their intersection, we have also

$$m\,(1 - \cos\theta') = \frac{p}{1 - \cos(\theta' - a)}.$$

The former of these conditions is satisfied if

$$\theta' + (\theta' - a) = 180^\circ,$$

since in this case it reduces to

$$1 - \cos^2\theta' = \sin^2\theta'.$$

The value of θ', as thus determined, is

$$\theta' = 90^\circ + \tfrac{1}{2}a,$$

and this value, substituted in the second equation of condition, yields

$$\begin{aligned} p &= m(1 + \sin\tfrac{1}{2}a)(1 - \sin\tfrac{1}{2}a) \\ &= m\cos^2\tfrac{1}{2}a. \end{aligned}$$

Hence the equation

$$r = \frac{m\cos^2\frac{1}{2}a}{1 - \cos(\theta - a)}$$

represents a parabola meeting the cardioid at right angles: and any assigned value of a determines one parabola. But in any such parabola, when $\theta = 180^\circ$, $r = \frac{1}{2}m$, independently of the value of a, since $\dfrac{\cos^2\frac{1}{2}a}{1 + \cos a} = \dfrac{1}{2}$; hence all of these parabolas pass through a single point on the axis of the cardioid, as was to be shown. [*F. H. Loud.*]

17

In his work, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, p. 65, Grassmann says: "Lagrange führt in seiner *Méc. Anal.*, p. 14 der neuen Ausgabe, einen Satz von Varignon an, dessen er sich zur Verknüpfung der verschiedenen Principien der Statik bedient. * * * * Dieser Satz ist, wie sich sogleich zeigen wird, unrichtig."

In what way is this theorem incorrect as used by Todhunter and others?

[*Asaph Hall.*]

SOLUTION.

Varignon's theorem as stated by Lagrange at the place above referred to is substantially as follows: If from a point in the plane of a parallelogram perpendiculars be let fall on two contiguous sides and on the co-initial diagonal, then the product of the diagonal into its perpendicular is equal to the sum or to the difference of the products of the sides into their respective perpendiculars *according as the point lies outside or inside the parallelogram.* The final clause is not correct, and, as Grassmann has pointed out, should be: *according as the point lies outside or inside the two angular spaces containing the diagonal and its prolongation.*

Varignon himself, of course, gives his theorem correctly.* But the manner in which he treats it is very lengthy and cumbersome: the enunciation of the theorem alone occupies a whole folio page and the demonstration is broken up into as many special cases as there are different positions for the point, and illustrated by over a dozen different diagrams.

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*See his *Nouvelle Mecanique ou Statique*, etc. Paris, 1725. Vol. I. p. 84 foll.

Lagrange, too, must have had in his mind the correct form of the proposition and his words "hors du parallélogramme" are only a slip of the pen that may be easily pardoned. This will appear from the developments with which he follows it, where he remarks that the theorem is true if the sides and diagonal of the parallelogram are moved within the lines in which they lie.

Todhunter, in his *Analytical Statics* (4th ed., Lon. 1874), p. 53, states the theorem in the following simple and correct form: "The algebraical sum of the moments of two component forces with respect to any point in the plane containing the two forces is equal to the moment of the resultant of the two forces." Similarly in his *Mechanics for Beginners* (5th ed., Lon. 1880), p. 40–1. In Geo. M. Minchin's *Statics* (2nd ed., Oxf. 1880), p. 67–8, we find: "The virtual work of a force is equal to the sum of the virtual works of its components, rectangular or oblique."

In these forms of the theorem, as statically expressed, any ambiguity disappears and it becomes impossible to repeat Lagrange's incorrectness, simply because the algebraical sign of "moments" and "works" is fixed by definition and leaves no room for uncertainty.

This precision and simplicity of statement however may be reached just as well in the purely *geometrical* form of the proposition, which ought to be made the basis of its applications to mechanics. It is only necessary to adopt the notion of plane areas affected by sign, or, more correctly speaking, to define the area of a triangle as a vector quantity. It will then be readily seen that Varignon's theorem is equivalent to the fundamental proposition in the theory of trilinear co-ordinates $ap + bq + cr = 2\Delta$.* Hamilton, Grassmann, Moebius, and others have clearly shown the advantages to be derived from treating areas as vectors and "geometrical" products, and perhaps no other proposition of such an elementary character could so forcibly illustrate the superiority of modern geometrical methods. In this respect it will prove interesting to contrast Varignon's presentation, which must be conceded to be a fine example of strictly Euclidean geometry, with W. K. Clifford's simple and elegant treatment of the same theorem, by modern methods, in his admirable little treatise on Dynamics (Part I: *Kinematic*, Lon. 1878), p. 92 foll.

[*Alexander Ziwet.*]

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*See *Salmon's Higher Plane Curves*, Chap. I.